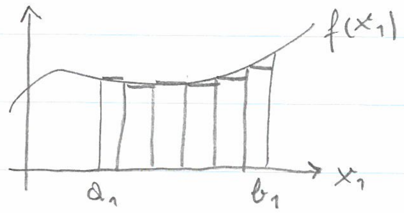
Mehrfachintegrale und Integralsätze

▷ **Wiederholung 1-dim. Integrale**

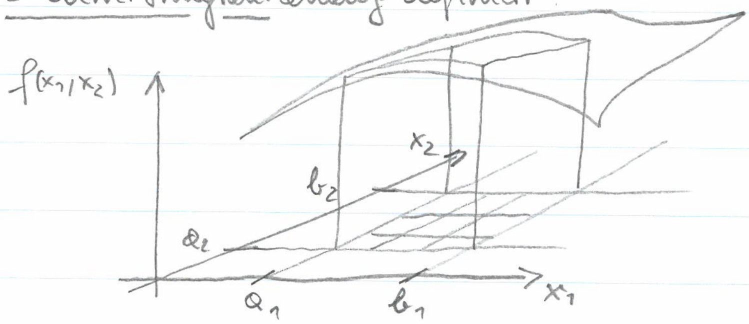


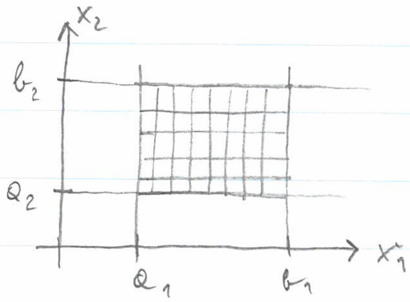
$$\int_{a_1}^{b_1} f(x_1)\,dx_1 = \lim_{\Delta x \to 0} \sum_i f(x_i)\,\Delta x$$

\# Falls $f(x_1)$ eine Länge: $\underset{m\ \cdot\ m}{f(x_1)\,dx_1} = m^2$ … Fläche unter dem Graphen zw. a_1 und b_1

\# Falls $f(x_1)$ eine Längendichte: $\underset{\frac{kg}{m}\ \cdot\ m}{f(x_1)\,dx_1} = kg$ … Masse des Stabs von a_1 bis b_1

▷ **2-dim. Integrale: analog definiert:**





$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1,x_2)\,\overbrace{dx_1\,dx_2}^{dA} := \int_{a_2}^{b_2}\left[\int_{a_1}^{b_1} f(x_1,x_2)\,dx_1\right]dx_2 \overset{\text{Theorem}}{=}$$

m^3 … $\quad m \quad m \cdot m$

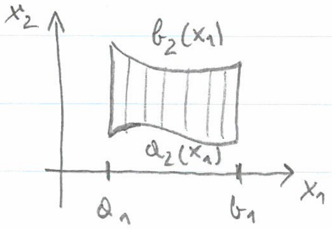
kg … $\quad \frac{kg}{m^2} \quad m \cdot m$

$$= \int_{a_1}^{b_1}\left[\int_{a_2}^{b_2} f(x_1,x_2)\,dx_2\right]dx_1 .$$

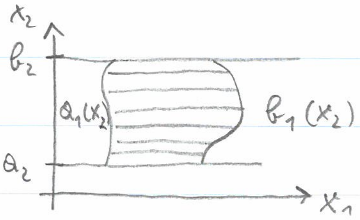
zuerst: inneres Integral … Fkt. (x_1)

dann: äusseres Integral … Zahl

Wie kann man über allgemeinere Integrationsbereiche als Rechtecke integrieren?

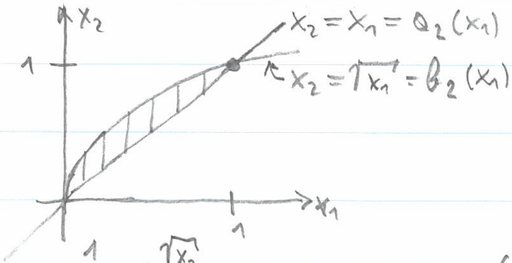


$$\rightarrow \int_{a_1}^{b_1}\left[\int_{a_2(x_1)}^{b_2(x_1)} f(x_1,x_2)\,dx_2\right]dx_1$$



$$\rightarrow \int_{a_2}^{b_2}\left[\int_{a_1(x_2)}^{b_1(x_2)} f(x_1,x_2)\,dx_1\right]dx_2$$

▷ Bsp:



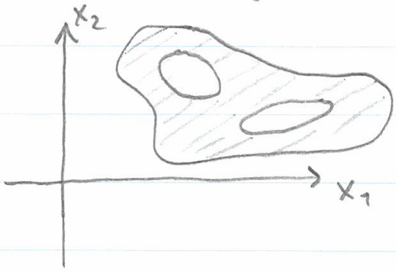
$$f(x_1,x_2) = x_1 x_2$$

$$\int_0^1 \int_{x_1}^{\sqrt{x_1}} x_1 x_2 \, dx_2 \, dx_1 = \int_0^1 x_1 \frac{x_2^2}{2}\Big|_{x_1}^{\sqrt{x_1}} dx_1 =$$

$$= \int_0^1 \frac{1}{2}x_1^2 - \frac{1}{2}x_1^3 \, dx_1 = \frac{1}{2}\frac{x_1^3}{3} - \frac{1}{2}\frac{x_1^4}{4}\Big|_0^1 =$$

$$= \frac{1}{6} - \frac{1}{8} = \frac{4-3}{24} = \underline{\underline{\frac{1}{24}}}.$$

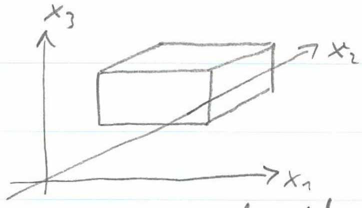
▷ Wie können noch allgemeinere Integrationsbereiche behandelt werden?



* Zerlegen in behandelbare Teilbereiche
* angepasste Koordinaten verwenden.

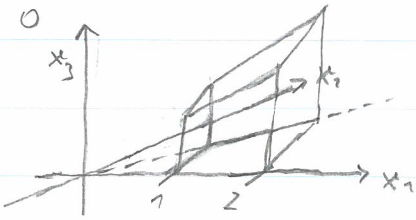
▷ **3-dim Integrale:** völlig analog

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1,x_2,x_3)\,\underbrace{dx_1\,dx_2\,dx_3}_{dV}$$

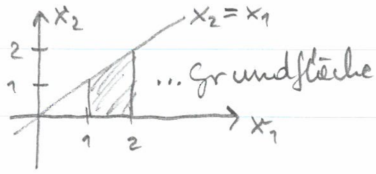


Integrationsbereich

Bsp:
$$\int_1^2\int_0^{x_1}\int_0^{x_1+x_2} x_2+x_3\,dx_3\,dx_2\,dx_1$$



$x_3 = x_1+x_2$ ... Deckebene



↳ **Teilintegrale:**

x_3-Integration:
$$\int_0^{x_1+x_2} x_2+x_3\,dx_3 = x_2x_3+\frac{x_3^2}{2}\Big|_0^{x_1+x_2} = x_2(x_1+x_2)+\frac{1}{2}(x_1+x_2)^2-0 =$$
$$= x_2^2+x_1x_2+\frac{1}{2}x_1^2+x_1x_2+\frac{1}{2}x_2^2 = \frac{1}{2}x_1^2+2x_1x_2+\frac{3}{2}x_2^2$$

x_2-Integration:
$$\int_0^{x_1}\frac{1}{2}x_1^2+2x_1x_2+\frac{3}{2}x_2^2\,dx_2 = \frac{1}{2}x_1^2x_2+2x_1\frac{x_2^2}{2}+\frac{3}{2}\frac{x_2^3}{3}\Big|_0^{x_1} =$$
$$= \frac{1}{2}x_1^3+x_1^3+\frac{1}{2}x_1^3 = 2x_1^3$$

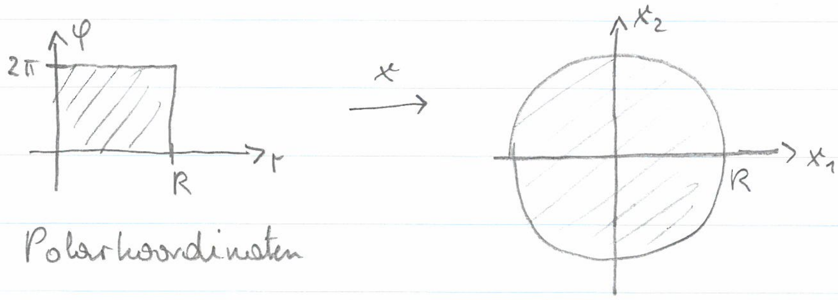
x_1-Integration:
$$\int_1^2 2x_1^3\,dx_1 = 2\frac{x_1^4}{4}\Big|_1^2 = \frac{1}{2}(16-1) = \frac{15}{2} = \underline{\underline{7{,}5}}.$$

▷ **Angepasste Koordinaten – Substitutionsformel:**

$$\underbrace{\iiint}_{x\text{-Grenzen}} \underbrace{f(x_1,x_2,x_3)}_{x}\,\underbrace{dx_1\,dx_2\,dx_3}_{\substack{\text{Volumen-}\\\text{element in}\\x\text{-Koordinaten}}} = \underbrace{\iiint}_{\substack{\text{entsprechende}\\p\text{-Grenzen}}} f(x(\underbrace{p_1,p_2,p_3}_{p}))\,|\det(x'(p))|\;dp_1\,dp_2\,dp_3$$

$|\det(x'(p))|\,dp_1\,dp_2\,dp_3$ ... Volumenelement in p-Koordinaten

▷ Bsp:



Polarkoordinaten

$$x(r,\varphi) = \begin{pmatrix} x_1(r,\varphi) \\ x_2(r,\varphi) \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

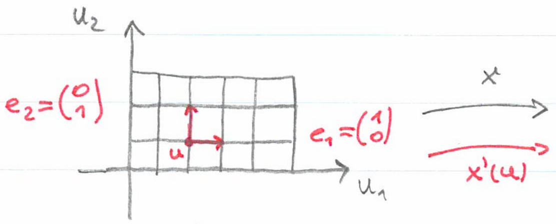
$$x'(r,\varphi) = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \varphi} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \varphi} \end{pmatrix}(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\det(x'(r,\varphi)) = \cos\varphi \cdot r\cos\varphi - \sin\varphi\cdot(-r\sin\varphi) = r(\cos^2\varphi + \sin^2\varphi) = r$$

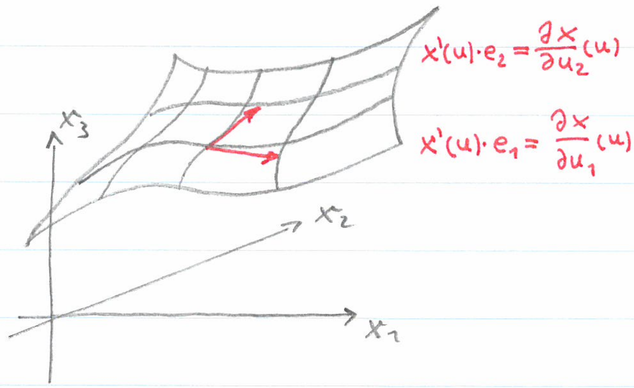
$$\text{Kreisfläche} = \iint\limits_{x\text{-Bereich}} 1\, dx_1\, dx_2 = \int_0^{2\pi}\int_0^R 1\cdot r\cdot dr\, d\varphi = \int_0^{2\pi} \frac{r^2}{2}\Big|_0^R d\varphi =$$

$$= \int_0^{2\pi} \frac{1}{2}R^2 d\varphi = \frac{1}{2}R^2\varphi\Big|_0^{2\pi} = \underline{\underline{R^2\pi}}.$$

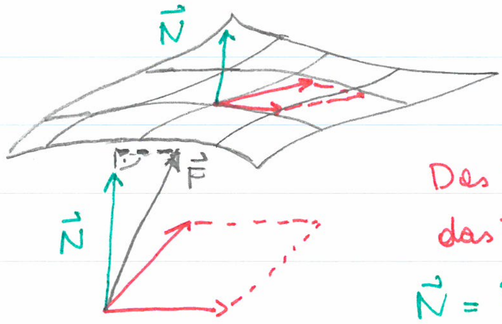
▷ Oberflächenintegrale



2-dim Parameterraum mit Parametern u_1 und u_2



$$x(u) = \begin{pmatrix} x_1(u) \\ x_2(u) \\ x_3(u) \end{pmatrix} \quad \ldots \text{ Parametrisierung der Oberfläche}$$



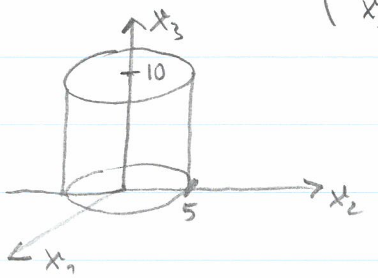
Das Parallelogramm approximiert das Flächenstück

$$\vec{N} = \frac{\partial x}{\partial u_1} \times \frac{\partial x}{\partial u_2}$$

$\vec{F} \cdot \vec{N}$ ... Fluss von $\vec{F}$ durch das Parallelogramm (inneres Produkt!)

$$\underline{\underline{\text{Gesamtfluss} = \iint \vec{F} \cdot d\vec{A}}} := \underline{\underline{\iint \vec{F} \cdot \vec{N} \, du_1 \, du_2}} = \underline{\underline{\int_{a_2}^{b_2} \int_{a_1}^{b_1} \vec{F}(x(u_1,u_2)) \cdot \vec{N}(u_1,u_2) \, du_1 \, du_2}}$$

▷ Bsp: Fluss von $\vec{F}(x) = \begin{pmatrix} x_2 \\ x_1 \\ x_3^2 \end{pmatrix}$ durch die Mantelfläche des Zylinders.

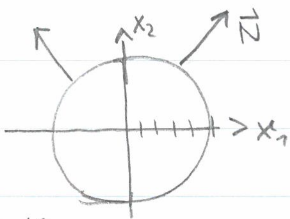


Parametrisierung (vgl. Zylinderkoordinaten):

$$x: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$(\varphi, z) \longmapsto x(\varphi, z) = \begin{pmatrix} 5\cos\varphi \\ 5\sin\varphi \\ z \end{pmatrix}$$

$$\frac{\partial x}{\partial \varphi} = \begin{pmatrix} -5\sin\varphi \\ 5\cos\varphi \\ 0 \end{pmatrix}, \quad \frac{\partial x}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$$\vec{N} = \frac{\partial x}{\partial \varphi} \times \frac{\partial x}{\partial z} = \begin{pmatrix} 5\cos\varphi \\ 5\sin\varphi \\ 0 \end{pmatrix}$$

$$\int_0^{2\pi} \int_0^{10} \begin{pmatrix} 5\sin\varphi \\ 5\cos\varphi \\ z^2 \end{pmatrix} \cdot \begin{pmatrix} 5\cos\varphi \\ 5\sin\varphi \\ 0 \end{pmatrix} dz \, d\varphi = \int_0^{2\pi} \int_0^{10} 25\sin\varphi\cos\varphi + 25\cos\varphi\sin\varphi \, dz \, d\varphi$$

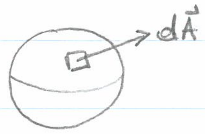
$$= \int_0^{2\pi} \int_0^{10} 25 \cdot \underbrace{2\cos\varphi\sin\varphi}_{\sin(2\varphi)} \, dz \, d\varphi = 25 \int_0^{2\pi} \sin(2\varphi) \cdot z \Big|_0^{10} d\varphi =$$

$$= 250 \int_0^{2\pi} \sin(2\varphi) \, d\varphi = 250 \, \frac{-\cos(2\varphi)}{2} \Big|_0^{2\pi} = 250\left(-\tfrac{1}{2} + \tfrac{1}{2}\right) = \underline{\underline{0}}$$

▷ Integralsätze von Gauß und Stokes in $\mathbb{R}^3$:

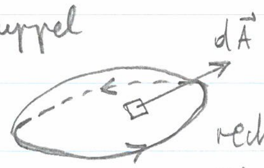
$$\underbrace{\iint}_{\text{geschlossene Oberfläche}} \vec{F}\cdot d\vec{A} = \underbrace{\iiint}_{\text{eingeschlossenes Volumen}} \operatorname{div}(\vec{F})\, dV$$ ... Satz von Gauß in $\mathbb{R}^3$

z.B. Kugel



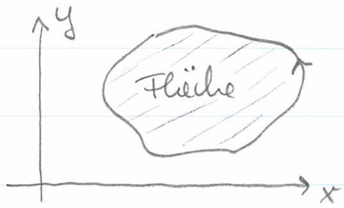
$$\underbrace{\int}_{\text{geschlossene Kurve}} \vec{F}\cdot d\vec{s} = \underbrace{\iint}_{\text{eingeschlossene Fläche}} \operatorname{rot}(\vec{F})\cdot d\vec{A}$$ ... Satz von Stokes in $\mathbb{R}^3$

z.B. Kuppel



▷ Satz von Green in $\mathbb{R}^2$:

$$\underbrace{\iint}_{\text{Fläche}} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\, dx\, dy = \underbrace{\int}_{\text{Randkurve}} f\, dx + g\, dy$$

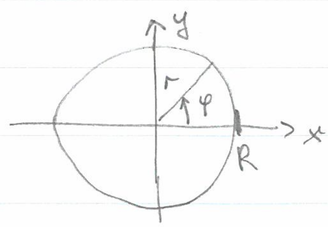


Beispiel: Flächenberechnung durch Wegintegral (= Arbeitsintegral)

Wähle $g(x,y) = x$ und $f(x,y) = 0$.

Satz von Green: $\underbrace{\iint}_{\text{Fläche}} 1 \cdot dx\, dy = \underbrace{\int}_{\text{Rand}} x\, dy$

Bsp. Kreis mit Radius r:



(a) $A = \iint\limits_{\text{Fläche}} dx\,dy$ in Polarkoordinaten parametrisiert

$x = r\cos\varphi$

$y = r\sin\varphi$

$$dx\,dy = \left|\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{pmatrix}\right| dr\,d\varphi$$

$$= \left|\det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi\end{pmatrix}\right| dr\,d\varphi$$

$$= r\,dr\,d\varphi$$

$$A = \int_0^{2\pi}\int_0^R r\,dr\,d\varphi = \int_0^{2\pi} \frac{R^2}{2}\,d\varphi = 2\pi\,\frac{R^2}{2} = \underline{\pi R^2}$$

(b) $A = \int\limits_{\text{Kreis}} x\,dy$ nach dem Winkel φ parametrisiert:

$x = R\cos\varphi \qquad dx = -R\sin\varphi\,d\varphi$

$y = R\sin\varphi \qquad dy = R\cos\varphi\,d\varphi$

$$A = \int_0^{2\pi} R\cos\varphi\, R\cos\varphi\,d\varphi = R^2 \int_0^{2\pi} \underbrace{\cos^2\varphi}_{=\frac{1}{2}(1+\cos(2\varphi))}\,d\varphi =$$

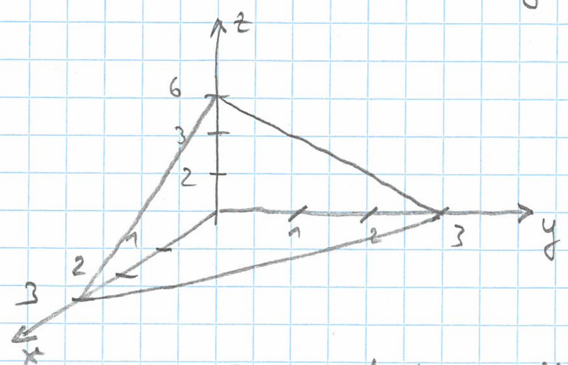
$$= R^2\,\frac{1}{2}\left[\varphi + \frac{1}{2}\sin(2\varphi)\right]\Big|_0^{2\pi} =$$

$$= \frac{R^2}{2}\,2\pi + 0 = \underline{\pi R^2}.$$

D <u>Beispiele zur Integralrechnung</u>

① Vektorfeld: $\vec{F}(x,y,z) = \begin{pmatrix} xy \\ -x^2 \\ x+z \end{pmatrix}$

Oberfläche: Teil der Ebene $2x + 2y + z = 6$ im ersten Oktanten

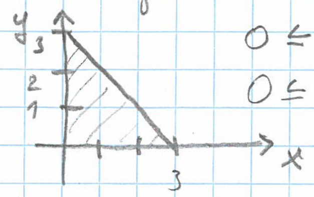


Gesucht: Fluss von $\vec{F}$ durch die Oberfläche.

Parametrisierung nach x und y:

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 6-2x-2y \end{pmatrix}$$

x und y frei wählbar in den Grenzen:



$0 \leq x \leq 3$

$0 \leq y \leq 3-x$

Normalvektor: $\frac{\partial X}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ $\quad \frac{\partial X}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$$\vec{N} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{Fluss} = \int_0^3 \int_0^{3-x} \vec{F} \cdot \vec{N} \, dy \, dx = \int_0^3 \int_0^{3-x} 2xy - 2x^2 + x + z \, dy \, dx =$$

$$= \int\int 2xy - 2x^2 + x + 6 - 2x - 2y \, dy \, dx =$$

$$= \int_0^3 \int_0^{3-x} 2xy - 2x^2 - x - 2y + 6 \, dy \, dx =$$

$$= \int_0^3 xy^2 - 2x^2 y - xy - y^2 + 6y \Big|_{y=0}^{y=3-x} dx =$$

$$= \int_0^3 x(3-x)^2 - 2x^2(3-x) - x(3-x) - (3-x)^2 + 6(3-x) \, dx =$$

$$= \int_0^3 9x - 6x^2 + x^3 - 6x^2 + 2x^3 - 3x + x^2 - 9 + 6x - x^2 + 18 - 6x \, dx =$$

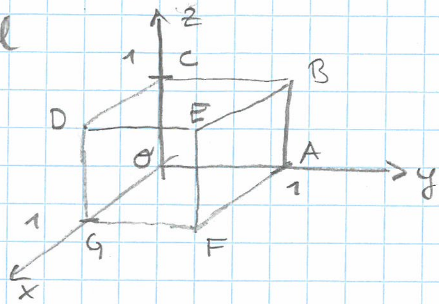
$$= \int_0^3 3x^3 - 12x^2 + 6x + 9 \, dx =$$

$$= 3\frac{x^4}{4} - 4x^3 + 3x^2 + 9x \Big|_0^3 =$$

$$= \frac{1}{4}\left[3 \cdot 81 - 16 \cdot 27 + 12 \cdot 9 + 4 \cdot 27\right] = \underline{\underline{\frac{27}{4}}}$$

② <u>Satz von Gauß:</u> $\vec{F} = \begin{pmatrix} 2x - z \\ x^2 y \\ -xz^2 \end{pmatrix}$ .... Vektorfeld

Raumgebiet: Einheitswürfel



Oberflächenintegrale:

$\iint\limits_{DEFG} \vec{F} \cdot d\vec{A}$: Parametrisierung nach y und z

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ y \\ z \end{pmatrix} \quad \text{für} \quad 0 \le y \le 1 \text{ und } 0 \le z \le 1$$

$$\frac{\partial X}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{\partial X}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$N = \frac{\partial X}{\partial y} \times \frac{\partial X}{\partial z} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ ... nach außen } \checkmark$$

$$\iint\limits_{DEFG} \vec{F} \cdot d\vec{A} = \int_0^1 \int_0^1 \begin{pmatrix} 2-z \\ y \\ -z^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dy\, dz =$$

$$= \int_0^1 \int_0^1 2 - z \; dy\, dz = \int_0^1 2y - zy \Big|_{y=0}^{1} dz =$$

$$= \int_0^1 2 - z \, dz = 2z - \frac{z^2}{2} \Big|_0^1 = 2 - \frac{1}{2} = \underline{1{,}5}$$

$\iint\limits_{ABCO} \vec{F} \cdot d\vec{A}$: Parametrisierung nach y und z

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}, \quad \frac{\partial X}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{\partial X}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Damit der Normalvektor nach aussen zeigt:

$$\vec{N} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\iint\limits_{ABCO} \vec{F} \cdot \vec{N} \, dy\, dz = \int_0^1 \int_0^1 \begin{pmatrix} -z \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} dy\, dz = \int_0^1 \int_0^1 z \, dy\, dz = \int_0^1 z \, dz =$$

$$= \frac{z^2}{2} \Big|_0^1 = \underline{0{,}5}$$

Analog: $\iint\limits_{ABEF} \vec{F}\cdot d\vec{A} = \ldots = 1/3$

$\iint\limits_{OGDC} \vec{F}\cdot d\vec{A} = \ldots = 0$

$\iint\limits_{BCDE} \vec{F}\cdot d\vec{A} = \ldots = -1/2$

$\iint\limits_{AFGO} \vec{F}\cdot d\vec{A} = \ldots = 0$

Summe aller Teiloberflächenintegrale $= \underline{\underline{\frac{11}{6}}}.$

Rechnung über Divergenz:

$$\mathrm{div}(\vec{F}) = \nabla\cdot\vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} 2x-z \\ x^2y \\ -xz^2 \end{pmatrix} =$$

$$= \frac{\partial}{\partial x}(2x-z) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(-xz^2) =$$

$$= \underline{2 + x^2 - 2xz}$$

$$\iiint\limits_{\text{Würfel}} \mathrm{div}(\vec{F})\, dV = \int_0^1\int_0^1\int_0^1 2 + x^2 - 2xz \, dx\, dy\, dz =$$

$$= \int_0^1\int_0^1 2x + \frac{x^3}{3} - x^2z \Big|_{x=0}^1 dy\, dz =$$

$$= \int_0^1\int_0^1 2 + \frac{1}{3} - z \quad dy\, dz =$$

$$= \int_0^1 \frac{7}{3}y - zy \Big|_{y=0}^1 dz =$$

$$= \int_0^1 \frac{7}{3} - z \, dz =$$

$$= \frac{7}{3}z - \frac{z^2}{2} \Big|_0^1 = \frac{7}{3} - \frac{1}{2} =$$

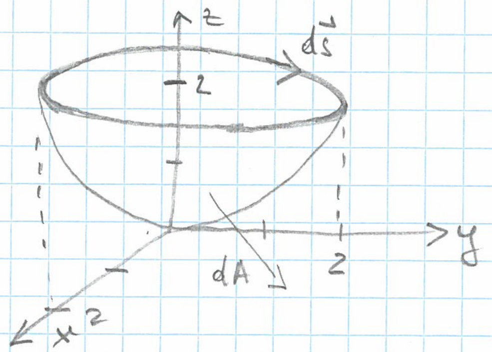
$$= \frac{14-3}{6} = \underline{\underline{\frac{11}{6}}}.$$

③ Satz von Stokes:

Vektorfeld $\vec{F} = \begin{pmatrix} 3y \\ -xz \\ yz^2 \end{pmatrix}$

Oberfläche: Paraboloid mit Gleichung $2z = x^2 + y^2$

bis zur Höhe $z = 2$



$\int\limits_{\text{Randkurve}} \vec{F} \cdot d\vec{s}$: Zylinderkoordinaten, Parametrisierung nach φ

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\cos\varphi \\ 2\sin\varphi \\ 2 \end{pmatrix} \qquad \begin{aligned} dx &= -2\sin\varphi \, d\varphi \\ dy &= 2\cos\varphi \, d\varphi \\ dz &= 0 \end{aligned}$$

$$\int\limits_{\text{Randkurve}} \vec{F} \cdot d\vec{s} = \int_{2\pi}^{0} \begin{pmatrix} 3 \cdot 2\sin\varphi \\ -2\cos\varphi \cdot 2 \\ 2\sin\varphi \cdot 2^2 \end{pmatrix} \cdot \begin{pmatrix} -2\sin\varphi \\ 2\cos\varphi \\ 0 \end{pmatrix} d\varphi =$$

$$= \int_{2\pi}^{0} -12\sin^2\varphi - 8\cos^2\varphi \, d\varphi =$$

$$= \int_{0}^{2\pi} 12\sin^2\varphi + 8\cos^2\varphi \, d\varphi = \int_{0}^{2\pi} 4 \cdot \sin^2\varphi + 8 \, d\varphi =$$

$$= \left(4 \cdot \frac{1}{2} + 8\right)\varphi \Big|_0^{2\pi} = 10 \cdot 2\pi = \underline{\underline{20\pi}}$$

$\iint\limits_{\text{Oberfläche}} \operatorname{rot}(\vec{F}) \cdot d\vec{A}$: $\operatorname{rot}(\vec{F}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} 3y \\ -xz \\ yz^2 \end{pmatrix} = \begin{pmatrix} z^2 + x \\ 0 - 0 \\ -z - 3 \end{pmatrix}$

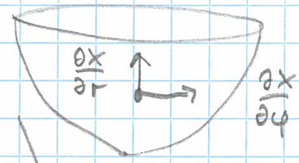
Parametrisierung der Oberfläche mit Zylinderkoordinaten:

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ \frac{1}{2}r^2 \end{pmatrix} \quad \text{mit} \quad \begin{aligned} 0 &\leq r \leq 2 \\ 0 &\leq \varphi \leq 2\pi \end{aligned}$$

$z = \frac{1}{2}(x^2 + y^2) = \frac{1}{2}r^2$

$$\frac{\partial X}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ r \end{pmatrix} \qquad \frac{\partial X}{\partial \varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}$$

Damit Normalvektor nach außen schaut:



$$\vec{N} = \frac{\partial x}{\partial \varphi} \times \frac{\partial x}{\partial r}$$

$$= \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ r \end{pmatrix} = \begin{pmatrix} r^2\cos\varphi \\ r^2\sin\varphi \\ -r\sin^2\varphi - r\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} r^2\cos\varphi \\ r^2\sin\varphi \\ -r \end{pmatrix}.$$

$$\iint_{\text{Oberfl.}} \operatorname{rot}(\vec{F}) \cdot d\vec{A} = \int_0^{2\pi}\int_0^2 \begin{pmatrix} \frac{1}{4}r^4 + r\cos\varphi \\ 0 \\ -\frac{1}{2}r^2 - 3 \end{pmatrix} \cdot \begin{pmatrix} r^2\cos\varphi \\ r^2\sin\varphi \\ -r \end{pmatrix} dr\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{4}r^6\cos\varphi + r^3\cos^2\varphi + \frac{1}{2}r^3 + 3r \;\; dr\, d\varphi =$$

$$= \int_0^{2\pi} \frac{1}{4}\frac{r^7}{7}\cos\varphi + \frac{r^4}{4}\cos^2\varphi + \frac{1}{2}\frac{r^4}{4} + 3\frac{r^2}{2} \Big|_{r=0}^{2} d\varphi =$$

$$= \int_0^{2\pi} \frac{32}{7}\cos\varphi + 4\cos^2\varphi + 2 + 6 \;\; d\varphi$$

$$= \frac{32}{7}\sin\varphi + 4\cdot\frac{1}{2}\varphi + 8\varphi \Big|_0^{2\pi} =$$

$$= 0 + 2\cdot 2\pi + 8\cdot 2\pi = \underline{\underline{20\pi}}.$$